Time-interval analysis of β decay

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Work on the event analysis of β decay [1] continued and resulted in the development of a novel method of beta-decay time-interval analysis that produces highly accurate results for the half-life, decay rate, and background rate, regardless of the event rate, nature of the detection-system dead time, and/or extent of the dead time, while being conceptually simple, free of notable numerical challenges, fast and exact. The method applies when event arrival times are measured and recorded individually, which can be accomplished easily using existing electronic modules, such as time-to-digital converter (TDC) or waveform digitizer (WFD). These modules are controlled by means of a personal computer, using customized software developed in our lab.

This report describes the principles behind the *time-interval analysis method* and demonstrates its robustness based on an example involving simulated events.

The half-life of a β-decaying nuclide is determined from the known time-dependence of the event rate expected under ideal conditions (i.e., in the absence of the detection system's dead time), which is described by a function we denoted by ρ and call the *ideal rate function*. For example, in the case of a single-component decay in the presence of a constant background *B*,

$$\rho = A \exp(-\lambda t) + B \,, \tag{1}$$

where *A* is the initial ideal rate (at time $t = 0$) in the absence of background, and

$$\lambda = \ln(2) / T_{1/2} \tag{2}$$

is the nuclide-specific decay constant, which is related to the nuclide's half-life $T_{1/2}$. Here it is assumed that the system's detection efficiency does not depend on ρ.

The time-interval analysis method was derived based on Eqs. (9) and (10) of Ref. [1], which express the probability d*p* that an event occurring at time zero will be followed by the next event in the time interval [t , t + dt), provided that a known detection-system dead time t_d follows the event detected at time zero:

$$\mathrm{d}p = \Theta(t - t_d) \exp[- <\rho>_d (t - t_d)] \, \rho_t \, \mathrm{d}t \,, \tag{3}$$

where

$$<\rho>_d = \frac{1}{t - t_d} \int_{t_d}^{t} \rho(t') dt' \;; \tag{4}$$

ρ_t is the value of ρ at time *t*; and $\Theta(t - t_d)$ is the Heaviside (unit-step) function. The Heaviside function reflects the fact that the probability of detecting an event in the time interval [0, t_d) equals zero.

Consequently, dp on the left-hand side of Eq. (3) is also the probability of detecting an event in time interval [t, t + dt) following the detection of no events in the time interval [t_d, t). In fact, replacing t_d in Eqs. (3) and (4) with $t - t_l$, where t_l is the detection-system live time preceding the detection of the event at time t, yields

$$dp = \Theta(t_l)\,\exp[-\langle\rho\rangle_l\, t_l]\,\rho_t\,dt\,, \qquad (5)$$

where

$$<\rho>_l = \frac{1}{t_l}\int_{t-t_l}^{t}\rho(t')dt' \qquad (6)$$

.

The key element in the time-interval analysis method, which ensures that the method is exact (i.e., not involving any approximations in its concept), is to know t_d exactly. Unfortunately, in reality, the actual value of t_d cannot be determined exactly for each measured event. However, if the timing of each measured event is recorded, it is possible to impose, by means of software, a known fixed *extendable* dead time τ_e that follows each measured event. The measured (primary) events that are not eliminated by the imposed dead time can then be used to form a secondary event set. If τ_e is set to be equal to (or greater than) the largest actual dead time t_d in the original (primary) event set, then the actual dead time t_d, as well as the actual live time t_l, for each event in the secondary set can be determined exactly. Consequently, an exact analysis can be performed on the secondary event set, even though the nature and/or the extent of the detection-system dead time may not be known exactly for each event in the primary set.

To ensure that the dead time τ_e imposed on the events recorded in a real measurement is sufficiently large but not too large (to avoid removing too many events), analysis of the primary-event set must be performed several times, each time with a different value of τ_e. By plotting the results of the analyses as a function of τ_e, it should be straight-forward to determine the best value of τ_e (here denoted by τ_m), below which the results show a trend, and above which the results vary randomly. The extent of the random variations for $\tau_e > \tau_m$ must be smaller than those normally expected based on the number of events analyzed. This is because the secondary event sets obtained from the same primary event set by imposing different values of τ_e are not statistically independent.

The goal of data analysis is to determine the best estimates (or most-likely values) of the parameters of ρ and their uncertainties. In the time-interval analysis method, as applied to a single measurement of beta decay that started at time t = 0, this is accomplished by evaluating quantity Z, which is proportional to the probability of obtaining, in a repeated measurement under the same conditions, the actual time sequence of events that survived after the dead time τ_m had been imposed. Then

$$Z = \exp(-\int_{t_z}^{t_f}\rho dt)\prod_{i=1}^{N}(dp/dt)_i \qquad (7)$$

,

where

$$(dp/dt)_i = \Theta[t_l(i)]\exp[-\int_{t_i - t_l(i)}^{t_i} \rho(t')dt']\rho_i \quad ; \tag{8}$$

N is the total number of such events; t_i (i = 1, 2, …, N) is the instant when the i-th event occurred; $t_l(i)$ is the detection-system live time period *preceding* event i; t_f is the instant when the measurement ended; ρ_i is the value of the ideal event rate at time t_i; and

$$t_z = \min(t_f\, , t_N + \tau_m)\ . \tag{9}$$

The exponential function in Eq. (7) represents the probability of measuring no events in the time interval (t_N , t_f).

Finally, the parameters of ρ [i.e., A, $T_{1/2}$, and B, if ρ is assumed to be given by Eqs.(1) and (2)] are varied iteratively in order to find the set of their most-likely values, which are taken to be those that maximize the value of Z. For practical reasons, this is done by minimizing the value of quantity E given by

$$E = -2\ln Z\ . \tag{10}$$

If several measurements are analyzed simultaneously so that, for example, a common value of $T_{1/2}$ can be determined, the parameters of ρ are varied iteratively in order to maximize the *product* of Z-values or to minimize the *sum* of E-values obtained for each measurement. The uncertainty of any parameter of ρ is obtained as the square root of the corresponding diagonal element of the inverse of the Hessian matrix of E.

Accuracy and statistical consistency of the results can be best assessed by applying the time-interval analysis method to simulated event sets that have been constructed based on imposed values for the parameters of ρ. The simulated event sets used to test the time-interval analysis method were made to mimic those obtained in the actual measurements of the ^{26m}Al half-life, which used the K-500 superconducting cyclotron, the Momentum Achromatic Recoil Separator, and the Precision On-Line Decay Facility at Texas A&M University [2]. Specifically, it was assumed that ρ is given by Eqs. (1) and (2), with B =1 s^{-1} and $T_{1/2}$ = 6.3452 s [3], while A ranged from 10^2 s^{-1} to 10^5 s^{-1}. The original sets of simulated events were made assuming that there is no dead time, but a dead time per event (τ_m) of up to 512 μs, as needed, was imposed on the data by the software before the beginning of the time-interval analysis. The total number of events in each primary set was about 60 million, which corresponds to a statistical precision slightly above 0.01%. The number of individual decay measurements in each simulated event set ranged from 66 at A = 10^5 s^{-1} to 57,693 at A = 10^2 s^{-1}. Each simulated measurement was assumed to last 125 s, which corresponds to about 20 half-lives.

In order to distinguish between the values of B, $T_{1/2}$, and A, on which the event simulation was based, and the corresponding values obtained in the analysis of the simulated event sets, lowercase symbols a, $t_{1/2}$, and b will be used for the latter.

To assess the meaningfulness and quality of results from the time-interval analysis method, a 500-channel decay spectrum was constructed for each simulated measurement, along with the corresponding spectrum of predicted values. These predicted values were obtained for each channel by integration of the most-likely ideal event rate ρ (as obtained in the analysis) over time, from the channel lower limit to the channel upper limit, while skipping the time intervals within the channel that were covered by the dead time. The corresponding channel contents of the individual spectra from the same set were then combined to construct a single spectrum in order to present statistically more meaningful results and to amplify and expose any systematic errors that might have occurred in the data analysis. An example of a spectrum and the results obtained this way are shown in Fig. 1.

Fig. 1 demonstrates that the time-interval analysis method produces accurate results, in particular

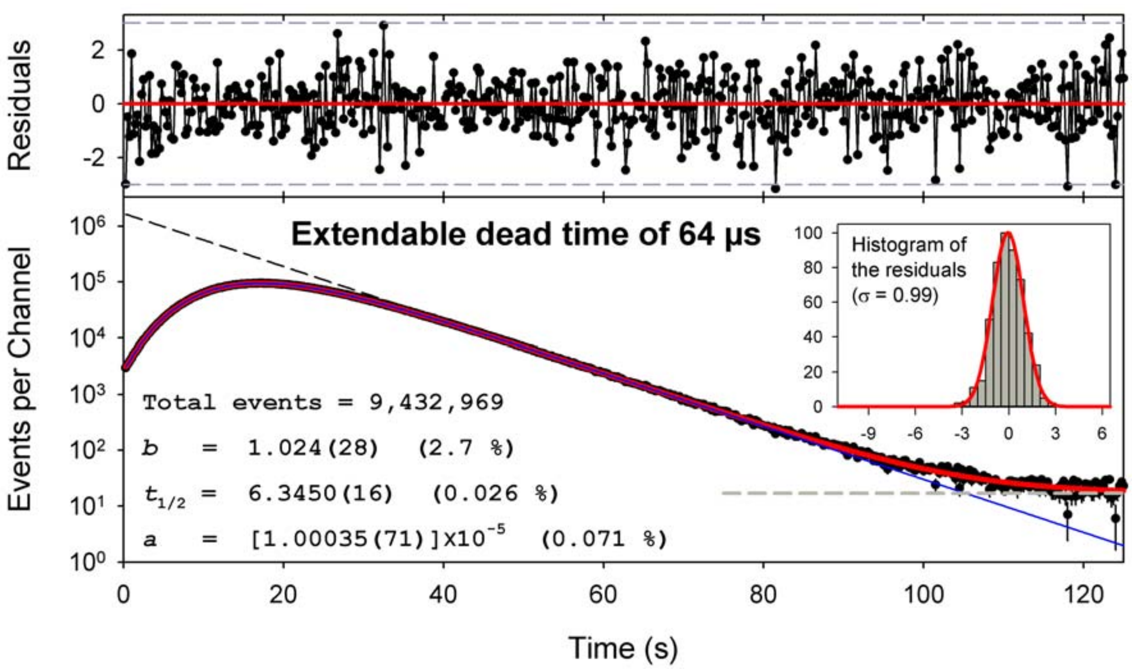

FIG. 1. Results of the time-interval analysis presented in the form of the combined decay spectrum and the corresponding spectrum of the residuals for the case of a simulated ideal event set obtained assuming $A = 10^5$ s^{-1}, $B = 1$ s^{-1}, and $T_{1/2} = 6.3452$ s, on which an extendable dead time of 64 µs was imposed. In the decay spectrum, the data points represent the number of events in each (0.25 s wide) channel, the thick solid (red) lines represent the expected values calculated based on the best estimates of the ideal rate parameters obtained in the analysis (and on the imposed dead time). Likewise, the thick dashed (gray) lines represent the background, while the thin solid (blue) lines represent the decay component. The thin dashed (black) lines represent the expected results under ideal conditions (i.e., no dead time). The residuals are shown as a function of time in a separate graph located above the corresponding graph of the decay spectrum, while their histograms are shown as inserts in the decay spectrum graph, using grey bars. Each histogram of the residuals was fitted by a Gaussian function. The best fit is shown by the solid (red) line and the best-fit standard deviation (σ) is indicated in the graph

for $t_{1/2}$, even in the case in which the decay spectrum is drastically distorted due to the presence of an *extendable* dead time. Note that the example shown in Fig. 1 is rather extreme and was chosen only to demonstrate the robustness of the time-interval analysis method.

[1] V. Horvat and J. C. Hardy, *Progress in Research*, Cyclotron Institute, Texas A&M University (2011-2012), p. V-28.
[2] http://cyclotron.tamu.edu/
[3] J.C. Hardy and I. S. Towner, Phys. Rev. C **79**, 055502 (2009).
[4] V. Horvat and J.C. Hardy, Nucl. Instrum. Methods Phys. Res. **A713**, 19 (2013).